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Accurate determination of diffuse view factors between planar surfaces

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Abstract—The contour double integral formula for the view factor between a pair of finite surfaces is a particularly simple formula to implement numerically. This paper suggests a method to improve the accuracy of the numerical results using this formula, both for non-intersecting surfaces, and for intersecting surfaces. In the latter case particularly, significant improvements in accuracy are achieved using the procedure outlined in the paper.

INTRODUCTION

In the thermal analysis of enclosures with non-participating intervening media and radiatively interacting surfaces, the surface radiative properties play an important role. The diffuse approximation is the most commonly used engineering approximation and assumes that the surfaces radiate diffusely. The surface reflectance model normally chosen is the diffuse or diffuse-specular model [1]. In such an analysis, the view factor (VF) plays a key role, since it indicates the proportion of diffuse radiation leaving a surface which reaches some other specified surface by direct radiative transport.

Although diffuse VFs represent an approximation to real directional behavior, it would be erroneous to assume that we could work with inaccurate values of VFs. In enclosures such errors would become magnified, as pointed out by Feingold [2]. Feingold used a simple example to illustrate this and concluded: "This should dispose of the idea that because real surfaces do not adhere exactly to Lambert's law, and because of some other simplifying assumptions which are normally made in radiant-interchange calculations, we can also afford the luxury of working with grossly inexact configuration factors."

It would be ideal if analytical solutions existed for VFs between any pair of planar diffuse surfaces. This, however, is not possible except for a few simple configurations which are listed in a number of references (e.g. ref. [1]). The vast majority of VF computations are done numerically. Toups [3] used the Nusselt projection method and obtained some reasonably accurate VFs for simple test configurations. Chung and Kim [4] used the Finite Element Method (FEM) to evaluate VFs, and illustrated this by evaluating the VF between adjoining plates. As we point out, this is an abnormal case and hence Chung's values are significantly in error. Incidentally, as pointed out by Feingold [2], the so-called analytic values of Hamilton and Morgan [5], quoted by Chung and Kim [4], are themselves in considerable error.

Sparrow [6], in an interesting paper, mathematically reduced the standard quadruple integral (double area integral) VF formula to the contour double integral formula (CDIF). Minning [7] used the CDIF to obtain a closed form expression for the VF between parallel ring sectors sharing a common center line. Shapiro [8] compared the CDIF and the area integral method and concluded that the CDIF was significantly more accurate than the area integral method. Mathsiak [9]. obtained an efficient algorithm based on the CDIF for determining VFs for plane polygonal areas. McAdam *et al.* [10] used the CDIF formula for determining shape factors applicable to greenhouses. In spite of these applications, it appears, from a survey of the literature, that there has been no attempt to use the CDIF for 'accurate' determination of VFs between surfaces with arbitrary relative orientations. The motivation for the present study was indeed a desire to determine VFs accurate to at least five significant digits and this has been accomplished.

VIEW FACTORS BY CDIF

Sparrow [6] has shown that the VF between two surfaces is given by

$$F_{1-2} = \frac{1}{2\pi A_1} \int_{c_1} \int_{c_2} \ln (S)\, \mathbf{dr}_1 \cdot \mathbf{dr}_2, \qquad (1)$$

where c_1 and c_2 represent the contours bounding the surfaces A_1 and A_2 respectively, $\mathbf{dr}_1$ and $\mathbf{dr}_2$ represent elemental lengths on the contours c_1 and c_2 respectively, and 'S' is the distance between these elements. The VF is evaluated numerically using formula (1), by replacing the double integral with double summation using quadratures. The desired accuracy is achieved

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NOMENCLATURE

$A, B, C, \ldots,$	constants
A_i	area of ith surface
C_i	ith contour
D	distance between contours
F_{i-j}	view factor from surface i to surface j
h_i	step size during ith trapezoidal integration
$\hat{i}, \hat{j}$	x and y direction unit vectors
L	length of the side
$\mathbf{r}_i$	position vector of the ith contour
S	distance between elements on different contours
T	Romberg/modified Romberg matrix.

Greek symbols

δ, ϕ	x and y increments along contour
ε	error in the CDIF.

Subscripts

c	common edge
s	side of polygon
sh	shortest side of either contour.

by the use of the Romberg method [11]. The following two configurations need different procedures to be adopted and hence are considered separately. These are

(a) the two surfaces do no intersect;
(b) the two surfaces intersect, thus sharing a common edge.

Non-intersecting contours

The Romberg method is well known and consists of obtaining an estimate for the value of an integral in the limit as the step size tends to zero. For the present calculations, the procedure consists of obtaining trapezoidal rule estimates with step lengths h_i, $h_i/2$, $h_i/4, \ldots,$ on either contour. Then a matrix $T(m,n)$ is created, where m is the column number and n is the row number. The Romberg extrapolation formula is [11]

$$T(m,n) = T(m-1,n+1) + \{T(m-1,n+1) - T(m-1,n)\}/(4^m - 1). \quad (2)$$

$T(i, n)$ corresponds to $T(h_n)$, where $h_n = h_i/2^{(n-1)}$. The elements of $T(m,n)$ represent subsequent extrapolations, with $T(n, 1)$ being the best extrapolation corresponding to $T(i,n)$. The order of magnitude of the error is h_n^{2n} and, thus, for small h_i, convergence is assured.

The above method is applied to the case of VFs between non-intersecting planar polygonal contours, for which a computer program in Fortran 77 was developed. The case of VFs calculated using this program, between the ends of a cylinder with regular polygonal cross-section, are compared with those obtained 'analytically' by Feingold [2] (these represent the most accurate values reported in the literature) in Table 1. The present calculations were all performed using double precision arithmetic using a Siemens main frame computer. For the triangle, square and pentagon, the present results compare favorably with Feingold's (i.e. up to five significant figures). For the hexagon and octagon, the present values differ considerably from Feingold's for the cases shown in italic. The values calculated using the method outlined in the present work can be guaranteed to at least six significant figures, as will be shown later. The error must lie in Feingold's values since his values were obtained using VF algebra. Thus, his values were exposed to the same error he had cautioned against. To validate the values obtained as described above, the VF from an octagon to the side was calculated. The side is a rectangle with one edge common with a side of the octagon, and height equal to the distance between the octagons. Since the rectangle and octagon intersect, the VF between them was evaluated using the method to be presented later in the paper. The octagon–octagon VF was then calculated using VF algebra. The results concurred up to the fifth decimal place (Table 1). Error in subsequent places is due to the lower precision of the VF between the intersecting surfaces, namely the octagon and the rectangle.

As a typical example, the Romberg matrix for the octagon case is shown in Table 2. For this case h_i is taken as $L_s/5$, where L_s is the length of a side of the figure. Six trapezoidal integrations were carried out based on the CDIF. The subsequent extrapolations are so efficient that trapezoidal integration need only be carried out for h_i, $h_i/2$ and $h_i/4$ in this case, further calculation being redundant for seven significant figure accuracy. It was found that for regular polygons, no more than five trapezoidal integrations were needed to obtain the stated accuracy with $h_i = L_s/5$.

This technique can also be applied to planar curved contours whose contour equation is known. However, the contour must be discretized with equal sized elements by the procedure given below, in order to use the Romberg procedure.

Discretization of an arbitrary planar curved contour $f(x, y) = 0$

The contour is discretized (in a coordinate system where the x, y plane contains the contour) into elements of length h_n as follows. At some starting point on the contour we draw a circle of radius h_n.

Table 1

Q	Case 1		Case 2		Case 3		Case 4		Case 5		By VF
	Present	Ref. [2]	Present	Ref. [2]	Present	Ref. [2]	Present	Ref. [2]	Present	Ref. [2]	algebra†
0.1	0.001374	0.001370	0.003162	0.003162	0.005416	0.005416	0.008135	0.008134	*0.014913*‡	*0.015087*	0.014912
0.2	0.005441	0.005439	0.012404	0.012404	0.020981	0.020978	0.031043	0.031042	*0.054899*	*0.054984*	0.054909
0.4	0.020955	0.020953	0.046137	0.046137	0.074812	0.074812	0.105660	0.105661	*0.169481*	*0.169516*	0.169480
0.6	0.044478	0.044476	0.093362	0.093362	0.143801	0.143802	0.193186	0.193186	*0.283565*	*0.283589*	0.283568
1.0	0.105606	0.105604	0.199825	0.199825	0.280463	0.280465	0.348576	0.346849§	*0.455013*	*0.455024*	0.455016
2.0	0.266561	0.266560	0.415253	0.415253	0.510752	0.510751	0.578373	0.578372	0.668799	0.668802	
4.0	0.483911	0.483910	0.632036	0.632036	0.708480	0.708480	0.756931	0.756932	0.816274	0.816276	
6.0	0.605148	0.605148	0.732584	0.732584	0.792843	0.792844	0.829490	0.829491	0.872973	0.872974	
10.0	0.731646	0.731645	0.826995	0.826994	0.868708	0.868709	0.893177	0.893177	0.921425	0.921425	
20.0	0.850521	0.850520	0.907853	0.907853	0.931325	0.931324	0.944679	0.944679	0.959744	0.959743	

Case 1, triangle; case 2, square; case 3, pentagon; case 4, hexagon; case 5, octagon (all are regular polygons as indicated in the text). L_s = length of side of polygon; D = distance between polygons; $Q = L_s/D$. All values correct to six significant figures.

† VFs evaluated using the octagon to rectangle VF (see text).

‡ Italic entries show excessive error in Feingold's calculation.

§ Suspected typographical error in Feingold's values.

The point of intersection of this circle with the contour and its center form the end points of the element. The initial h_i value must be about one-fifth the smallest radius of curvature of the contour so as to accurately discretize it.

The equation of the discretizing circle centered at a point (a, b) on the contour is

$$(x-a)^2+(y-b)^2 = h_n^2. \tag{3}$$

Let the tangent to the contour at (a, b) intersect the discretizing circle at (x_1, y_1) and let $\delta_1 = x_1 - a$, $\phi_1 = y_1 - b$. Since the tangent line will intersect the circle at two diametrically opposite points, (δ_1, ϕ_1) are chosen such that $(\nabla f \times \{\delta_1 \hat{i} + \phi_1 \hat{j}\})$ is always the same sign, so that the discretizing circle creeps over the contour in one direction only (see Fig. 1). Let (δ_1, ϕ_1) be the initial solution for the point of intersection of the circle with the contour, and let (δ_2, ϕ_2) be a better approximation such that:

$$\left.\begin{aligned} \delta_2 &= \delta_1 + g \\ \phi_2 &= \phi_1 + k \end{aligned}\right\}, \tag{4}$$

where g and k are very small compared to δ_1 and ϕ_1 respectively. Thus by equation (3):

$$(\delta_1+g)^2+(\phi_1+k)^2 = h_n^2. \tag{5}$$

Now by a Taylor series expansion:

$$f(a+\delta_2, b+\phi_2) = f(a,b)+\delta_2\, \partial f/\partial x|_{(a,b)} + \phi_2\, \partial f/\partial y|_{(a,b)} + \text{higher order terms.}$$

Assuming that the point specified by equation (4) lies on the contour, and $f(a, b) = 0$, we would have:

$$\{g\, \partial f/\partial x + k\, \partial f/\partial y\}|_{(a,b)} = -\{\delta_1\, \partial f/\partial x + \phi_1\, \partial f/\partial y\}|_{(a,b)}. \tag{6}$$

Solving equations (5) and (6) simultaneously, g and k are obtained, from which we get δ_2 and ϕ_2. Replacing (δ_1, ϕ_1) with (δ_2, ϕ_2), we can continue this process until convergence. Assuming that $(m-1)$ iterations are needed, we obtain (δ_m, ϕ_m), and the next point on the contour will be $(a+\delta_m, b+\delta_m)$.

This method is useful for second and higher order curves. Linear contours (i.e. polygons) can be discretized in a much more straightforward manner by taking equal sized steps along a side starting from a corner.

As a test case, the above method of discretization coupled with the Romberg procedure was used to evaluate the VF between co-axial circles of equal diameter, a distance D apart, that circumscribe equilateral triangles of side L. Table 3 shows these calculations for a wide range of values of the ratio L/D. The exact value is obtained from the formula given in ref. [1]. It appears that Feingold [2] has rounded his results to six decimal places. There is a remarkably good agreement (at least six significant digits) between the present calculation and the exact values.

Table 2. Romberg matrix for view factor between regular octagons of equal size

0.6707516 $T(1,1)$	0.6687968 $T(2,1)$	0.6687994 $T(3,1)$	0.6687994 $T(4,1)$	0.6687994 $T(5,1)$	0.6687994 $T(6,1)$
0.6692855 $T(1,2)$	0.6687993 $T(2,3)$	0.6687994 $T(3,2)$	0.6687994 $T(4,2)$	0.6687994 $T(5,2)$	
0.6689208 $T(1,3)$	0.6687994 $T(2,3)$	0.6687994 $T(3,3)$	0.6687994 $T(4,3)$		
0.6688298 $T(1,4)$	0.6687994 $T(2,4)$	0.6687994 $T(3,4)$			
0.6688070 $T(1,5)$	0.6687994 $T(2,5)$				
0.6688013 $T(1,6)$					

Side to distance ratio = 2. h_i = (length of side)/5. $T(m,n)$ are the Romberg matrix values, where $T(m,n)$ is given by equation (2).

Intersecting contours

Two planar contours always intersect on a straight line (Fig. 2). It is clear from the CDIF that when $S = 0$ (i.e. along the common edge), there will be a logarithmic singularity in the integrand (i.e. ln (0)).

Equation (1) is evaluated by moving along the respective contours c_1 and c_2. While moving along the common edge on both contours we are bound to encounter the singularity. A simple formula is given below for the CDIF along any straight common edge.

Clearly the CDIF along the common edge is (see Fig. 3)

$$\varepsilon = -(1/\{2\pi A_1\}) \int_{x_1=0}^{L_c} \int_{x_2=0}^{L_c} \ln|x_1 - x_2| \, dx_1 \, dx_2. \quad (7)$$

This is integrated analytically to yield

$$\varepsilon = -(1/\{2\pi A_1\}) L_c^2 \{\ln(L_c) - 1.5\}. \quad (8)$$

To find the VF between two intersecting contours we use the previously mentioned Romberg method. However, the integral along the common edge is not evaluated numerically. The basic values ($T(h)$ in Table 4) represent the CDIF, excluding the integral along the common edge. The correction ε is then added to the best Romberg extrapolation based on $T(h_i)$, $T(h_i/2)$, etc. to get the required VF.

Consider the case of two square plates intersecting along an edge and having an included angle of 30°. The Romberg matrix for this case, along with the correction ε, is shown in Table 4. This case was handled by Chung and Kim [4] using the FEM and

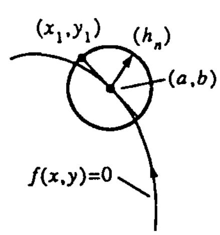

FIG. 1. A curved contour with the discretizing circle drawn thereon.

Table 3. Comparison of VFs between equal parallel circles as described in the text

L/D	Present study	Feingold [2]	Exact [1]
0.1	0.003311295	0.003315	0.003311295
1.0	0.2087122	0.208712	0.2087121
2.0	0.4312707	0.431271	0.4312707
10.0	0.8411466	0.841147	0.8411466

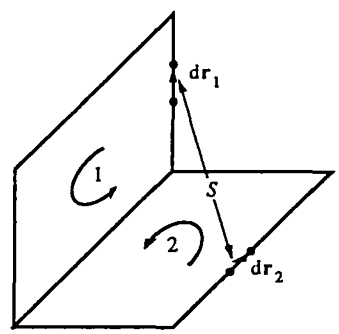

FIG. 2. An example of adjoining planar polygonal plates. Arrows show the direction of travel along respective contours.

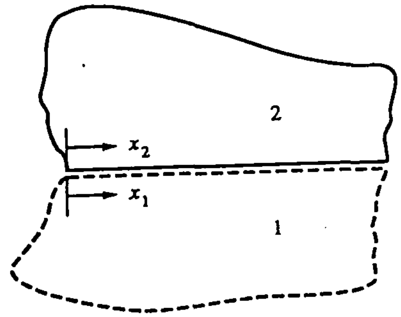

FIG. 3. Common edge showing the variables x_1 and x_2.

Table 4. Romberg matrix for the VF between two adjoining square plates with an included angle of 30°

0.7475610	0.7468307	0.7467771	0.7467669	0.7467644	0.7467639
0.7470133	0.7467805	0.7467670	0.7467645	0.7467639	
0.7468387	0.7467679	0.7467645	0.7467639		
0.7467856	0.7467647	0.7467639			
0.7467699	0.7467639				
0.7468654					

$L_s = 10, h_i = 1$. Correction $\varepsilon = -0.1277353$. The correction ε is added to $T(6, 1)$ to get a VF of 0.619029.

Table 5. Comparisons of VFs determined by various methods between two adjoining square plates at various included angles to each other

Author (method)	Angle (°) 30	60	90	120	150
Feingold [2] (Exact)	**0.619028**	0.370905	0.200044	0.086615	0.021346
Present (CDIF)	**0.619029**	0.370906	0.200044	0.086615	0.021345
Hamilton and Morgan [5]	**0.6202**	0.3712	0.20004	0.0870	0.0215
Toups [3] (Nusselt projection method—60 × 60 grid)	**0.61878**			0.08662	
Chung and Kim [4] (FEM—40 × 40 mesh)	**0.68786**	0.38133	0.20255	0.08729	0.02147

they obtained a value of 0.68786 with a 40 × 40 mesh (see bold faced entries in Table 5). However, Feingold [2] obtained a value of 0.619029 for the same problem, using the analytical formulae derived for this case by Hamilton and Morgan [5]. Using the CDIF coupled with the Romberg procedure discussed above, a value of 0.619029 was obtained with $h_i = L_c/10$ and six basic integrations. The sixth basic integration covered 1280 elements along both contours. Contrast this with the 1600 (40 × 40) area elements in Chung's case. Not only is the present method general (it will evaluate the VF between any two planar contours whether or not they are intersecting), it is also far simpler to implement than an FEM routine. In Table 5 a comparison is also given with the data of Hamilton and Morgan [5] and Toups [3] for various included angles between two adjoining square plates. In all cases the present calculations are seen to be in excellent agreement with the values given in ref. [2], which may be taken as a basis for comparison since they are obtained from an analytically derived formula. In view of this the results obtained by the other authors quoted are all in error to different extents. The FEM value of Chung seems to be the worst.

APPLICATION TO A SAMPLE ENCLOSURE

In order to demonstrate further the usefulness of the present CDIF formulation, we present below a VF algebra analysis for the enclosure shown in Fig. 4. The top to bottom VF (i.e. F_{5-6}) was calculated using $h_i = 1$. Six trapezoidal integrations were carried out. It was found that with just $T(h_i)$ and $T(h_i/2)$, a convergence to seven significant figures was obtained, further calculations being redundant. The VF for this case is 0.6698614. For the top to side case we obtained a VF of F_{5-1} (or F_{5-2}, F_{5-3}, F_{5-4}) = 0.082534 using the method applicable to intersection contours. Here six significant figures were taken because a comparison of $T(5, 1)$ and $T(6, 1)$ showed agreement only up to the fifth place. A study of the best extrapolations after the fourth, fifth and sixth (i.e. $T(4, 1)$, $T(5, 1)$ and $T(6, 1)$, respectively) basic integrations suggests that further basic integration coupled with the Romberg procedure would cause a change of ± 1 in the sixth decimal place. By VF algebra the top to side VF is (1.0000000–0.6698614)/4 = 0.0825346. We note here that by the two different methods we get a VF of 0.08253 if we round off to five decimal places. Considering six decimal places we get a value of

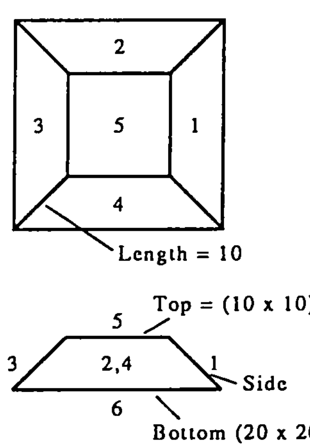

FIG. 4. A sample enclosure consisting of six sides. The VFs include those for non-intersecting as well as intersecting areas.

$F_{5-1} = 0.082534 \pm 1.0 \times 10^{-6}$. Clearly an idea of the accuracy of the VF can be gleaned from observation of the Romberg matrix.

CONCLUSION

It is clear that evaluating the VF using the CDIF with the trapezoidal rule coupled with Romberg extrapolation yields very accurate values. The method is very simple to implement and an error estimate can be made by observing the best extrapolates, i.e. $T(1,1)$, $T(2,1)$, $T(3,1)$, etc. The method is convergent [11] with diminishing step size.

A simple formula giving the value of the CDIF along the intersecting line was derived for intersecting areas. The VFs so calculated compare well with values computed using VF algebra. The method is also capable of giving very accurate values of the VF for areas bounded by curved contours.

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